

CANONICAL WEIERSTRASS REPRESENTATION OF MINIMAL AND MAXIMAL SURFACES IN THE THREE-DIMENSIONAL MINKOWSKI SPACE

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ABSTRACT. We prove that any minimal (maximal) strongly regular surface in the three-dimensional Minkowski space locally admits canonical principal parameters. Using this result, we find a canonical representation of minimal strongly regular time-like surfaces, which makes more precise the Weierstrass representation and shows more precisely the correspondence between these surfaces and holomorphic functions (in the Gauss plane). We also find a canonical representation of maximal strongly regular space-like surfaces, which makes more precise the Weierstrass representation and shows more precisely the correspondence between these surfaces and holomorphic functions (in the Lorentz plane). This allows us to describe locally the solutions of the corresponding natural partial differential equations.

1. INTRODUCTION

In [2] we proved that any minimal strongly regular surface in Euclidean space can be endowed locally with canonical principal parameters. Using this fact, in [1] we found a canonical Weierstrass representation of minimal strongly regular surfaces.

In this paper we consider strongly regular surfaces in the three-dimensional Minkowski space \mathbb{R}_1^3 .

We prove that any minimal strongly regular time-like surface can be endowed locally with canonical principal parameters. Using this result we prove the following

Theorem 1. (Canonical Weierstrass representation of minimal time-like surfaces) *Any minimal strongly regular time-like surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D} \subset \mathbb{C}$ parameterized with canonical principal parameters has locally a representation of the type*

$$\begin{aligned}(z_1)_x - i(z_1)_y &= \frac{1}{2} \frac{w^2 + 1}{w'}, \\ (z_2)_x - i(z_2)_y &= -\frac{i}{2} \frac{w^2 - 1}{w'}, \\ (z_3)_x - i(z_3)_y &= -\frac{w}{w'},\end{aligned}$$

where $w = u(x, y) + iv(x, y)$ is a holomorphic function in \mathbb{C} satisfying the conditions

$$\begin{aligned}u^2 + v^2 - 1 &\neq 0, \quad \mu := \frac{(u_x^2 + u_y^2)}{(u^2 + v^2 - 1)^2}; \\ \mu &> 0, \quad \mu_x \mu_y \neq 0.\end{aligned}$$

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As a consequence of this theorem we obtain a local description of the solutions of the natural partial differential equation of minimal time-like surfaces.

Further we apply the above scheme for maximal strongly regular space-like surfaces in \mathbb{R}_1^3 .

We prove that any maximal strongly regular space-like surface admits locally canonical principal parameters. Using this theorem we prove

Theorem 2. (Canonical Weierstrass representation of maximal space-like surfaces) *Any maximal strongly regular space-like surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D} \subset \mathbb{L}$ parameterized with canonical principal parameters has locally a representation of the type*

$$\begin{aligned} (z_1)_x + j(z_1)_y &= -\frac{w}{w'} \\ (z_2)_x + j(z_2)_y &= \frac{1}{2} \frac{w^2 - 1}{w'}, \\ (z_3)_x + j(z_3)_y &= \frac{j}{2} \frac{w^2 + 1}{w'}, \end{aligned}$$

where $w = u(x, y) + jv(x, y)$ is a holomorphic function in \mathbb{L} satisfying the conditions

$$\begin{aligned} u^2 - v^2 + 1 &\neq 0, \quad \mu := \frac{(u_x^2 - u_y^2)}{(u^2 - v^2 + 1)^2}; \\ \mu &> 0, \quad \mu_x \mu_y \neq 0. \end{aligned}$$

As a consequence of this theorem we obtain a local description of the solutions of the natural partial differential equation of maximal space-like surfaces.

2. MINIMAL TIME-LIKE SURFACES AND CANONICAL PRINCIPAL PARAMETERS

Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a space-like surface in hyperbolic space \mathbb{R}_1^3 and ∇ be the flat Levi-Civita connection of the standard metric in \mathbb{R}_1^3 . The unit normal vector field to \mathcal{M} is denoted by l and E, F, G ; e, f, g stand for the coefficients of the first and the second fundamental forms, respectively. In this case we have

$$E = \mathbf{z}_x^2 > 0, \quad G = \mathbf{z}_y^2 > 0, \quad EG - F^2 > 0, \quad l^2 = -1.$$

We suppose that the surface has no umbilical points and the principal lines on \mathcal{M} form a parametric net, i.e.

$$F(x, y) = f(x, y) = 0, \quad (x, y) \in \mathcal{D}.$$

Then the principal curvatures ν_1, ν_2 and the principal geodesic curvatures (geodesic curvatures of the principal lines) γ_1, γ_2 are given by

$$(2.1) \quad \nu_1 = \frac{e}{E}, \quad \nu_2 = \frac{g}{G}; \quad \gamma_1 = -\frac{E_y}{2E\sqrt{G}}, \quad \gamma_2 = \frac{G_x}{2G\sqrt{E}}.$$

We consider the tangential frame field $\{X, Y\}$ determined by

$$X := \frac{\mathbf{z}_x}{\sqrt{E}}, \quad Y := \frac{\mathbf{z}_y}{\sqrt{G}}$$

and suppose that the moving frame XYl is positive oriented so that

$$\nu_1 - \nu_2 > 0.$$

Then the following Frenet type formulas for the frame field XYl are valid

$$\begin{aligned}
(2.2) \quad \nabla_X X &= \gamma_1 Y - \nu_1 l, & \nabla_Y X &= \gamma_2 Y, \\
\nabla_X Y &= -\gamma_1 X, & \nabla_Y Y &= -\gamma_2 X - \nu_2 l, \\
\nabla_X l &= -\nu_1 X, & \nabla_Y l &= -\nu_2 Y.
\end{aligned}$$

The Codazzi equations have the following form

$$(2.3) \quad \gamma_1 = \frac{Y(\nu_1)}{\nu_1 - \nu_2} = \frac{(\nu_1)_y}{\sqrt{G}(\nu_1 - \nu_2)}, \quad \gamma_2 = \frac{X(\nu_2)}{\nu_1 - \nu_2} = \frac{(\nu_2)_x}{\sqrt{E}(\nu_1 - \nu_2)}$$

and the Gauss equation can be written as follows:

$$X(\gamma_2) - Y(\gamma_1) + \gamma_1^2 + \gamma_2^2 = \nu_1 \nu_2 = K,$$

or

$$\frac{(\gamma_2)_x}{\sqrt{E}} - \frac{(\gamma_1)_y}{\sqrt{G}} + \gamma_1^2 + \gamma_2^2 = \nu_1 \nu_2 = K.$$

A time-like surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ parameterized with principal parameters is said to be *strongly regular* if (cf [2])

$$\gamma_1(x, y)\gamma_2(x, y) \neq 0, \quad (x, y) \in \mathcal{D}.$$

The Codazzi equations (2.3) imply that

$$\gamma_1 \gamma_2 \neq 0 \iff (\nu_1)_y (\nu_2)_x \neq 0.$$

Then the following formulas

$$(2.4) \quad \sqrt{E} = \frac{(\nu_2)_x}{\gamma_2(\nu_1 - \nu_2)} > 0, \quad \sqrt{G} = \frac{(\nu_1)_y}{\gamma_1(\nu_1 - \nu_2)} > 0$$

are valid for strongly regular surfaces. Because of (2.4) formulas (2.2) become

$$\begin{aligned}
X_x &= \frac{\gamma_1 (\nu_2)_x}{\gamma_2 (\nu_1 - \nu_2)} Y - \frac{\nu_1 (\nu_2)_x}{\gamma_2 (\nu_1 - \nu_2)} l, \\
Y_x &= -\frac{\gamma_1 (\nu_2)_x}{\gamma_2 (\nu_1 - \nu_2)} X, \\
l_x &= -\frac{\nu_1 (\nu_2)_x}{\gamma_2 (\nu_1 - \nu_2)} X; \\
X_y &= \frac{\gamma_2 (\nu_1)_y}{\gamma_1 (\nu_1 - \nu_2)} Y, \\
Y_y &= -\frac{\gamma_2 (\nu_1)_y}{\gamma_1 (\nu_1 - \nu_2)} X - \frac{\nu_2 (\nu_1)_y}{\gamma_1 (\nu_1 - \nu_2)} l, \\
l_y &= -\frac{\nu_2 (\nu_1)_y}{\gamma_1 (\nu_1 - \nu_2)} Y.
\end{aligned}$$

and the fundamental Bonnet theorem for strongly regular time-like surfaces states as follows:

Theorem 2.1. *Given four functions $\nu_1(x, y)$, $\nu_2(x, y)$, $\gamma_1(x, y)$, $\gamma_2(x, y)$ defined in a neighborhood \mathcal{D} of (x_0, y_0) satisfying the following conditions*

$$\begin{aligned} 1) \quad & \nu_1 - \nu_2 > 0, \quad \gamma_1(\nu_1)_y > 0, \quad \gamma_2(\nu_2)_x > 0, \\ 2.1) \quad & \left(\ln \frac{(\nu_1)_y}{\gamma_1} \right)_x = \frac{(\nu_1)_x}{\nu_1 - \nu_2}, \quad \left(\ln \frac{(\nu_2)_x}{\gamma_2} \right)_y = -\frac{(\nu_2)_y}{\nu_1 - \nu_2}, \\ 2.2) \quad & \frac{\nu_1 - \nu_2}{2} \left(\frac{(\gamma_2^2)_x}{(\nu_2)_x} - \frac{(\gamma_1^2)_y}{(\nu_1)_y} \right) + (\gamma_1^2 + \gamma_2^2) = \nu_1 \nu_2 \end{aligned}$$

and an initial positive oriented orthonormal frame $\mathbf{z}_0 X_0 Y_0 l_0$.

Then there exists a unique strongly regular time-like surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}_0$ ($(x_0, y_0) \in \mathcal{D}_0 \subset \mathcal{D}$) with prescribed invariants $\nu_1, \nu_2, \gamma_1, \gamma_2$ such that

$$z(x_0, y_0) = z_0, \quad X(x_0, y_0) = X_0, \quad Y(x_0, y_0) = Y_0, \quad l(x_0, y_0) = l_0.$$

Similarly to the case of minimal surfaces in Euclidean space we shall prove that any minimal strongly regular surface admits locally geometric principal parameters. We can assume that $\nu_1 > 0$ and we refer to the function $\nu = \nu_1$ as the normal curvature function.

Proposition 2.2. *Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a minimal strongly regular surface whose parametric net is principal. Then there exist locally principal parameters (\bar{x}, \bar{y}) such that*

$$\mathbf{z}_{\bar{x}}^2 = \mathbf{z}_{\bar{y}}^2 = \frac{1}{\nu}, \quad \nu = \nu_1 > 0.$$

Proof: Taking into account (2.1) and (2.3), we obtain

$$(\ln \sqrt{\nu E})_y = 0, \quad (\ln \sqrt{\nu G})_x = 0,$$

which shows that νE is only a function of x and νG is only a function of y .

Let $(x_0, y_0) \in \mathcal{D}$. We introduce new parameters (\bar{x}, \bar{y}) in a neighborhood of (x_0, y_0) by the formulas

$$\bar{x} = \int_{x_0}^x \sqrt{\nu E} dx + \bar{x}_0, \quad \bar{y} = \int_{y_0}^y \sqrt{\nu G} dy + \bar{y}_0,$$

where \bar{x}_0 and \bar{y}_0 are constants. It follows immediately that (\bar{x}, \bar{y}) are again principal parameters and satisfy the required property. \square

We call the parameters from the above proposition *canonical principal* parameters.

Further we assume that the minimal strongly regular time-like surface \mathcal{M} under consideration is parameterized with canonical principal parameters.

We use the following notations:

$$\nu = \nu_1 > 0, \quad \nu_2 = -\nu < 0, \quad \nu_1 - \nu_2 = 2\nu > 0.$$

Further we have

$$(2.5) \quad E = G = \frac{1}{\nu}, \quad e = -g = 1, \quad \gamma_1 = (\sqrt{\nu})_x, \quad \gamma_2 = -(\sqrt{\nu})_y.$$

Then the equalities (2.2) become

$$\begin{aligned}
 X_x &= \frac{\nu_y}{2\nu} Y - \sqrt{\nu} l, \\
 Y_x &= -\frac{\nu_y}{2\nu} X, \\
 l_x &= -\sqrt{\nu} X; \\
 X_y &= -\frac{\nu_x}{2\nu} Y, \\
 Y_y &= \frac{\nu_x}{2\nu} X + \sqrt{\nu} l, \\
 l_y &= +\sqrt{\nu} Y.
 \end{aligned}
 \tag{2.6}$$

and the integrability conditions of (2.6) reduce to

$$\Delta \ln \nu - 2\nu = 0.
 \tag{2.7}$$

Theorem 2.1 applied to minimal strongly regular time-like surfaces parameterized with natural principal parameters implies:

Corollary 2.3. *Given a function $\nu(x, y) > 0$ in a neighborhood \mathcal{D} of (x_0, y_0) with $\nu_x \nu_y \neq 0$, satisfying the equation (2.7) and an initial positive oriented orthonormal frame $\mathbf{z}_0 X_0 Y_0 l_0$.*

Then there exists a unique minimal strongly regular time-like surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}_0$ ($(x_0, y_0) \in \mathcal{D}_0 \subset \mathcal{D}$), for which

- (i) *(x, y) are canonical principal parameters;*
- (ii) *the invariants $\nu_1, \nu_2, \gamma_1, \gamma_2$ are the following functions*

$$\nu_1 = \nu, \quad \nu_2 = -\nu, \quad \gamma_1 = (\sqrt{\nu})_y, \quad \gamma_2 = -(\sqrt{\nu})_x;$$

- (iii) *$\mathbf{z}(x_0, y_0) = \mathbf{z}_0$, $X(x_0, y_0) = X_0$, $Y(x_0, y_0) = Y_0$, $l(x_0, y_0) = l_0$.*

Further we refer to (2.7) as the *natural partial differential equation* of minimal time-like surfaces.

The above statement gives a one-to-one correspondence between minimal strongly regular time-like surfaces (considered up to a motion) and the solutions of the natural partial differential equation, satisfying the conditions

$$\nu > 0, \quad \nu_x \nu_y \neq 0.
 \tag{2.8}$$

3. CANONICAL WEIERSTRASS REPRESENTATION OF MINIMAL STRONGLY REGULAR TIME-LIKE SURFACES

Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a minimal strongly regular time-like surface parameterized with canonical principal parameters.

Equalities (2.6) imply the following formulas for the Gauss map $l = l(x, y)$ ($l^2 = -1$):

$$\begin{aligned}
 l_{xx} &= \frac{\nu_x}{2\nu} l_x - \frac{\nu_y}{2\nu} l_y + \nu l, \\
 l_{xy} &= \frac{\nu_y}{2\nu} l_x + \frac{\nu_x}{2\nu} l_y, \\
 l_{yy} &= -\frac{\nu_x}{2\nu} l_x + \frac{\nu_y}{2\nu} l_y + \nu l
 \end{aligned}
 \tag{3.1}$$

and the normal vector function $l(x, y)$ satisfies the partial differential equation:

$$\Delta l - 2\nu l = 0.$$

The next statement makes precise the properties of the Gauss map of a minimal time-like surface in terms of the canonical principal parameters.

Proposition 3.1. *Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a minimal strongly regular time-like surface parameterized with canonical principal parameters. Then the Gauss map $l = l(x, y)$, $(x, y) \in \mathcal{D}$; $l^2 = -1$ has the following properties:*

$$(3.2) \quad l_x^2 = l_y^2 = \nu > 0, \quad l_x l_y = 0, \quad \nu_x \nu_y \neq 0.$$

Conversely, if a vector function $l(x, y)$, $l^2 = -1$ has the properties (3.2), then there exists locally a unique (up to a motion) minimal strongly regular time-like surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$ determined by

$$(3.3) \quad \mathbf{z}_x = -\frac{1}{\nu} l_x, \quad \mathbf{z}_y = \frac{1}{\nu} l_y,$$

so that (x, y) are geometrical principal parameters and $\nu(x, y)$ is the normal curvature function of \mathcal{M} .

Proof: The equalities $l_x = -\nu \mathbf{z}_x$, $l_y = \nu \mathbf{z}_y$, and (2.5) imply (3.2).

For the inverse, it follows immediately that (3.2) implies (3.1). Furthermore, the second equality of (3.1) implies that the system (3.3) is integrable.

Since (3.3) and (3.1) imply (2.6), it follows that $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$ is a minimal strongly regular time-like surface parameterized with canonical principal parameters with normal curvature function $\nu = l_x^2 = l_y^2 > 0$.

Furthermore, it follows that the function $\nu = l_x^2 = l_y^2$ satisfies the equation (2.7). \square

Thus, any minimal strongly regular time-like surface locally is determined by the system (3.3), where the vector function $l = l(x, y)$, $l^2 = -1$ satisfies the conditions (3.2). The so obtained minimal surface is parameterized with canonical principal parameters.

Now, let $H^2(-1) : \xi^2 + \eta^2 - \zeta^2 = -1$ be the unit time-like sphere centered at the origin O and $l(\xi, \eta, \zeta)$, $\zeta \neq 1$ be the position vector of an arbitrary point on $H^2(-1)$, different from the pole $(0, 0, 1)$. Let us denote by (x, y) the coordinates of any point in the parametric plane \mathbb{C} and consider the standard conformal map $(x, y) \rightarrow (\xi, \eta, \zeta)$ of $H^2(-1)$ given by

$$\begin{aligned} \xi &= \frac{2x}{x^2 + y^2 - 1}, \\ l : \quad \eta &= \frac{2y}{x^2 + y^2 - 1}, \quad x^2 + y^2 - 1 \neq 0. \\ \zeta &= -\frac{x^2 + y^2 + 1}{x^2 + y^2 - 1}; \end{aligned}$$

The vector function $l = l(x, y)$ ($l^2 = -1$) has the properties

$$l_x^2 = l_y^2 = \frac{4}{(x^2 + y^2 - 1)^2}, \quad l_x l_y = 0$$

and generates the Enneper time-like surface.

The function

$$\nu = \frac{4}{(x^2 + y^2 - 1)^2} \quad x^2 + y^2 - 1 \neq 0$$

satisfies the equation (2.7).

Now we make more precise the Weierstrass representation of minimal strongly regular time-like surfaces, stated in Theorem 1.

Proof of Theorem 1

Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y), (x, y) \in \mathcal{D} \subset \mathbb{C}$ be a minimal strongly regular surface parameterized with canonical principal parameters. Since the Gauss map $l = l(x, y)$ ($l^2 = -1$) of \mathcal{M} is conformal and the orthogonal frame field $l_x l_y l$ is left oriented, then the vector function l is given locally by

$$(3.4) \quad \begin{aligned} \xi &= \frac{2u(x, y)}{u^2(x, y) + v^2(x, y) - 1}, \\ l : \quad \eta &= \frac{2v(x, y)}{u^2(x, y) + v^2(x, y) - 1}, \quad u^2(x, y) + v^2(x, y) - 1 \neq 0, \\ \zeta &= -\frac{u^2(x, y) + v^2(x, y) + 1}{u^2(x, y) + v^2(x, y) - 1}; \end{aligned}$$

where $w = u(x, y) + i v(x, y)$ is a holomorphic function in \mathbb{C} .

We denote

$$\mu = \frac{(u_x^2 + u_y^2)}{(u^2 + v^2 - 1)^2}.$$

If ν is the normal curvature function of \mathcal{M} , then the vector function $\mathbf{z} = \mathbf{z}(x, y)$ satisfies the system

$$(3.5) \quad \begin{aligned} \mathbf{z}_x &= -\frac{1}{\nu} l_x = -\frac{1}{\nu} (u_x l_u + v_x l_v), \\ \mathbf{z}_y &= \frac{1}{\nu} l_y = \frac{1}{\nu} (u_y l_u + v_y l_v), \end{aligned}$$

which implies that $\nu = \frac{4(u_x^2 + u_y^2)}{(u^2 + v^2 - 1)^2} = 4\mu$. Hence the holomorphic function w satisfies the conditions

$$u^2 + v^2 - 1 \neq 0; \quad \mu > 0, \quad \mu_x \mu_y \neq 0.$$

Denoting by $w' = \frac{dw}{dz} = \frac{\partial w}{\partial z} = u_x - i u_y$, we have

$$\frac{1}{w'} = \frac{u_x + i u_y}{u_x^2 + v_y^2}$$

and (3.5) can be written in the form

$$(3.6) \quad \mathbf{z}_x - i \mathbf{z}_y = -\frac{1}{w'} \frac{(u^2 + v^2 - 1)^2}{4} (l_u + i l_v).$$

Taking into account (3.4), we obtain from (3.6) the following formulas:

$$\begin{aligned}(z_1)_x - i(z_1)_y &= \frac{1}{2} \frac{w^2 + 1}{w'}, \\ (z_2)_x - i(z_2)_y &= -\frac{i}{2} \frac{w^2 - 1}{w'}, \\ (z_3)_x - i(z_3)_y &= -\frac{w}{w'},\end{aligned}$$

which proves the assertion. \square

As an application we obtain a corollary for the solutions of the natural partial differential equation.

Corollary 3.2. *Any solution ν of the natural partial differential equation (2.7) satisfying the condition (2.8) is given locally by the formula*

$$(3.7) \quad \nu = \frac{4(u_x^2 + u_y^2)}{(u^2 + v^2 - 1)^2}, \quad u^2 + v^2 - 1 \neq 0,$$

where $w = u + iv$ is a holomorphic function in \mathbb{C} .

Proof: Let $\nu(x, y)$ be a solution of (2.7) satisfying the conditions (2.8) and let us consider the minimal strongly regular time-like surface \mathcal{M} , generated by ν . According to Theorem 1 it follows that the normal curvature function ν of \mathcal{M} locally has the form (3.7). \square

It is a direct verification that any function ν , given by (3.7), where $w = u + iv$ ($u_x^2 + u_y^2 > 0$) is a holomorphic function in \mathbb{C} , satisfies (2.7).

Remark 3.3. The canonical Weierstrass representation of minimal strongly regular time-like surfaces is based on the Gauss map of the Enneper time-like surface ($w = z$). Choosing the Gauss map of any other minimal strongly regular time-like surface \mathcal{M} , we shall obtain its corresponding representation. This remark is also valid for the representation (3.7) of the solutions of the natural partial differential equation (2.7).

4. MAXIMAL SPACE-LIKE SURFACES AND CANONICAL PRINCIPAL PARAMETERS

Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a space-like surface in \mathbb{R}_1^3 . In this case we assume that

$$E = \mathbf{z}_x^2 > 0, \quad G = \mathbf{z}_y^2 < 0, \quad l^2 = 1.$$

We suppose that the surface has no umbilical points and the principal lines on \mathcal{M} form a parametric net, i.e.

$$F(x, y) = f(x, y) = 0, \quad (x, y) \in \mathcal{D}.$$

Then the principal curvatures ν_1, ν_2 and the principal geodesic curvatures (geodesic curvatures of the principal lines) γ_1, γ_2 are given by

$$(4.1) \quad \nu_1 = \frac{e}{E}, \quad \nu_2 = \frac{g}{G}; \quad \gamma_1 = \frac{E_y}{2E\sqrt{-G}}, \quad \gamma_2 = -\frac{G_x}{2\sqrt{E}G}.$$

We consider the tangential frame field $\{X, Y\}$ determined by

$$X := \frac{\mathbf{z}_x}{\sqrt{E}}, \quad Y := \frac{\mathbf{z}_y}{\sqrt{-G}}$$

and suppose that the moving frame XYl is always positive oriented so that $\nu_1 - \nu_2 > 0$.

The following Frenet type formulas for the frame field XYl are valid

$$\begin{aligned}
(4.2) \quad \nabla_X X &= \gamma_1 Y + \nu_1 l, & \nabla_Y X &= -\gamma_2 Y, \\
\nabla_X Y &= \gamma_1 X, & \nabla_Y Y &= -\gamma_2 X - \nu_2 l, \\
\nabla_X l &= -\nu_1 X, & \nabla_Y l &= -\nu_2 Y.
\end{aligned}$$

The Codazzi equations have the following form

$$(4.3) \quad \gamma_1 = -\frac{Y(\nu_1)}{\nu_1 - \nu_2}, \quad \gamma_2 = -\frac{X(\nu_2)}{\nu_1 - \nu_2}$$

and the Gauss equation can be written as follows:

$$X(\gamma_2) + Y(\gamma_1) + \gamma_1^2 - \gamma_2^2 = \nu_1 \nu_2 = K.)$$

A space-like surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ without umbilical points is said to be *strongly regular* if (cf [2])

$$\gamma_1(x, y)\gamma_2(x, y) \neq 0, \quad (x, y) \in \mathcal{D}.$$

Since

$$\gamma_1 \gamma_2 \neq 0 \iff (\nu_1)_y (\nu_2)_x \neq 0,$$

then the following formulas

$$(4.4) \quad \sqrt{E} = \frac{-(\nu_2)_x}{\gamma_2(\nu_1 - \nu_2)} > 0, \quad \sqrt{-G} = \frac{-(\nu_1)_y}{\gamma_1(\nu_1 - \nu_2)} > 0.$$

are valid for strongly regular surfaces. Because of (4.4) formulas (4.2) become

$$\begin{aligned}
X_x &= -\frac{\gamma_1(\nu_2)_x}{\gamma_2(\nu_1 - \nu_2)} Y - \frac{\nu_1(\nu_2)_x}{\gamma_2(\nu_1 - \nu_2)} l, \\
Y_x &= -\frac{\gamma_1(\nu_2)_x}{\gamma_2(\nu_1 - \nu_2)} X, \\
l_x &= \frac{\nu_1(\nu_2)_x}{\gamma_2(\nu_1 - \nu_2)} X; \\
X_y &= \frac{\gamma_2(\nu_1)_y}{\gamma_1(\nu_1 - \nu_2)} Y, \\
Y_y &= \frac{\gamma_2(\nu_1)_y}{\gamma_1(\nu_1 - \nu_2)} X + \frac{\nu_2(\nu_1)_y}{\gamma_1(\nu_1 - \nu_2)} l, \\
l_y &= \frac{\nu_2(\nu_1)_y}{\gamma_1(\nu_1 - \nu_2)} Y.
\end{aligned}$$

and the fundamental Bonnet theorem for strongly regular space-like surfaces states as follows:

Theorem 4.1. *Given four functions $\nu_1(x, y)$, $\nu_2(x, y)$, $\gamma_1(x, y)$, $\gamma_2(x, y)$; $(x, y) \in \mathcal{D}$ satisfying the following conditions:*

$$\begin{aligned} 1) \quad & \nu_1 - \nu_2 > 0, \quad \gamma_1(\nu_1)_y < 0; \quad \gamma_2(\nu_2)_x < 0, \\ 2.1) \quad & \left(\ln \frac{-(\nu_1)_y}{\gamma_1} \right)_x = \frac{(\nu_1)_x}{\nu_1 - \nu_2}, \quad \left(\ln \frac{-(\nu_2)_x}{\gamma_2} \right)_y = -\frac{(\nu_2)_y}{\nu_1 - \nu_2}; \\ 2.2) \quad & \frac{\nu_1 - \nu_2}{2} \left(\frac{(\gamma_1^2)_y}{(\nu_1)_y} + \frac{(\gamma_2^2)_x}{(\nu_2)_x} \right) - \gamma_1^2 + \gamma_2^2 + \nu_1 \nu_2 = 0. \end{aligned}$$

Then there exists a unique (up to a motion) strongly regular surface with prescribed invariants ν_1 , ν_2 , γ_1 , γ_2 .

A space-like surface is maximal if $H = 0$, i.e. $\nu_1 + \nu_2 = 0$. Next we introduce geometric principal parameters on maximal strongly regular space-like surfaces. We can assume that $\nu_1 > 0$ and we refer to the function $\nu = \nu_1$ as the normal curvature function.

Proposition 4.2. *Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a maximal strongly regular space-like surface whose parametric net is principal. Then there exist locally principal parameters (\bar{x}, \bar{y}) such that*

$$\mathbf{z}_{\bar{x}}^2 = -\mathbf{z}_{\bar{y}}^2 = \frac{1}{\nu}, \quad \nu = \nu_1 > 0.$$

Proof: Let (x_0, y_0) be a point in \mathcal{D} . Taking into account (4.1) and (4.3), we obtain

$$(\ln \sqrt{\nu E})_y = 0, \quad (\ln \sqrt{-\nu G})_x = 0,$$

which shows that νE is only a function of x and νG is only a function of y . Introducing new parameters (\bar{x}, \bar{y}) by the formulas

$$\bar{x} = \int_{x_0}^x \sqrt{\nu E} dx + \bar{x}_0, \quad \bar{y} = \int_{y_0}^y \sqrt{-\nu G} dy + \bar{y}_0$$

we obtain that (\bar{x}, \bar{y}) are again principal parameters and satisfy the required property. \square

We call the parameters from the above lemma *canonical principal* parameters.

From now on we assume that the maximal strongly regular space-like surface \mathcal{M} is parameterized with canonical principal parameters.

We use the following notations:

$$\nu = \nu_1 > 0, \quad \nu_2 = -\nu < 0, \quad \nu_1 - \nu_2 = 2\nu > 0.$$

Further we have

$$(4.5) \quad E = -G = \frac{1}{\nu}, \quad e = g = 1, \quad \gamma_1 = -(\sqrt{\nu})_y, \quad \gamma_2 = (\sqrt{\nu})_x.$$

Then the equalities (4.2) become

$$\begin{aligned}
 X_x &= -\frac{\nu_y}{2\nu} Y + \sqrt{\nu} l, \\
 Y_x &= -\frac{\nu_y}{2\nu} X, \\
 l_x &= -\sqrt{\nu} X; \\
 X_y &= -\frac{\nu_x}{2\nu} Y, \\
 Y_y &= -\frac{\nu_x}{2\nu} X + \sqrt{\nu} l, \\
 l_y &= \sqrt{\nu} Y.
 \end{aligned}
 \tag{4.6}$$

and the integrability conditions of (4.6) reduce to

$$(\ln \nu)_{xx} - (\ln \nu)_{yy} + 2\nu = 0
 \tag{4.7}$$

Theorem 4.1 applied to maximal strongly regular space-like surfaces parameterized with canonical principal parameters implies:

Corollary 4.3. *Given a function $\nu(x, y) > 0$ in a neighborhood \mathcal{D} of (x_0, y_0) with $\nu_x \nu_y \neq 0$, satisfying the equation (4.7) and an initial orthonormal frame $\mathbf{z}_0 X_0 Y_0 l_0$.*

Then there exists a unique strongly regular surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}_0$ ($(x_0, y_0) \in \mathcal{D}_0 \subset \mathcal{D}$), such that

- (i) *(x, y) are canonical principal parameters;*
- (ii) *$z(x_0, y_0) = z_0$, $X(x_0, y_0) = X_0$, $Y(x_0, y_0) = Y_0$, $l(x_0, y_0) = l_0$;*
- (iii) *\mathcal{M} is a maximal strongly regular space-like surface with invariants*

$$\nu_1 = \nu, \quad \nu_2 = -\nu, \quad \gamma_1 = -(\sqrt{\nu})_y, \quad \gamma_2 = (\sqrt{\nu})_x.$$

Further we refer to (4.7) as the *natural partial differential equation* of minimal space-like surfaces.

The above statement gives a one-to-one correspondence between maximal strongly regular space-like surfaces (considered up to a motion) and the solutions of the natural partial differential equation, satisfying the condition

$$\nu > 0, \quad \nu_x \nu_y \neq 0.
 \tag{4.8}$$

5. CANONICAL WEIERSTRASS REPRESENTATION OF MAXIMAL STRONGLY REGULAR SPACE-LIKE SURFACES

Let $H^2(1) : \xi^2 + \eta^2 - \zeta^2 = 1$ be the unit space-like sphere in \mathbb{R}_1^3 centered at the origin O and $l(\xi, \eta, \zeta)$ be the position vector of an arbitrary point on $H^2(1)$. If $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ is a space-like surface, then its Gauss map is $l : \mathcal{D} \longrightarrow H^2(1)$. Equalities (4.6) imply the

following formulas for the Gauss map:

$$\begin{aligned}
 l_{xx} &= \frac{\nu_x}{2\nu} l_x + \frac{\nu_y}{2\nu} l_y - \nu l, \\
 l_{xy} &= \frac{\nu_y}{2\nu} l_x + \frac{\nu_x}{2\nu} l_y, \\
 l_{yy} &= \frac{\nu_x}{2\nu} l_x + \frac{\nu_y}{2\nu} l_y + \nu l
 \end{aligned}
 \tag{5.1}$$

The vector function $l(x, y)$, $l^2 = 1$ satisfies the partial differential equation

$$l_{xx} - l_{yy} + 2\nu l = 0.$$

The next statement makes precise the properties of the Gauss map of a maximal space-like surface in terms of canonical principal parameters.

Proposition 5.1. *Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a maximal strongly regular space-like surface parameterized with canonical principal parameters. Then the Gauss map $l = l(x, y)$, $l^2 = 1$ has the following properties:*

$$l_x^2 = -l_y^2 = \nu > 0, \quad l_x l_y = 0, \quad \nu_x \nu_y \neq 0. \tag{5.2}$$

Conversely, if a vector function $l(x, y)$, $l^2 = 1$ has the properties (5.2), then there exists locally a unique (up to a motion) maximal strongly regular space-like surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$ determined by

$$\mathbf{z}_x = -\frac{1}{\nu} l_x, \quad \mathbf{z}_y = \frac{1}{\nu} l_y, \tag{5.3}$$

so that (x, y) are geometrical principal parameters and $\nu(x, y)$ is the normal curvature function of \mathcal{M} .

Proof: The equalities $l_x = -\nu \mathbf{z}_x$, $l_y = \nu \mathbf{z}_y$, and (4.5) imply (5.2).

For the inverse, it follows immediately that (5.2) implies (5.1). Furthermore, the second equality of (5.1) implies that the system (5.3) is integrable.

Since (5.3) and (5.1) imply (4.6), it follows that $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$ is a maximal strongly regular space-like surface parameterized with canonical principal parameters whose normal curvature function is ν .

Furthermore, it follows that the function $\nu = l_x^2 = -l_y^2$ satisfies the equation (4.7). \square

Thus, any maximal strongly regular space-like surface locally is determined by the system (5.3), where the vector function $l = l(x, y)$, $l^2 = 1$ satisfies the conditions (5.2). The so obtained maximal surface is parameterized with canonical principal parameters.

In the study of maximal space-like surfaces the role of the Gauss plane is played by the Lorentz plane $\mathbb{L} = \{z = x + jy\}$, $j^2 = -1$. The metric in the Lorentz plane is given by the quadratic form $x^2 - y^2$.

Let us denote by (x, y) the coordinates of any point in the parametric plane \mathbb{L} and consider the parametrization $(x, y) \rightarrow (\xi, \eta, \zeta)$ of $H^2(1)$ given by

$$(5.4) \quad \begin{aligned} \xi &= \frac{x^2 - y^2 - 1}{x^2 - y^2 + 1}, \\ l: \quad \eta &= \frac{2x}{x^2 - y^2 + 1}, \quad x^2 - y^2 + 1 \neq 0. \\ \zeta &= \frac{2y}{x^2 - y^2 + 1}, \end{aligned}$$

This vector function $l(x, y), l^2 = 1$ has the properties

$$l_x^2 = -l_y^2 = \frac{4}{(x^2 - y^2 + 1)^2}, \quad l_x l_y = 0$$

and generates the space-like Enneper surface.

The map (5.4) is conformal and the function

$$\frac{4}{(x^2 - y^2 + 1)^2} \quad x^2 - y^2 + 1 \neq 0$$

satisfies the equation (4.7).

Let $w = w(z)$ be a map in \mathbb{L} given by

$$w = u + jv : \quad \begin{aligned} u &= u(x, y), \\ v &= v(x, y). \end{aligned}$$

The differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are introduced as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - j \frac{\partial}{\partial y} \right).$$

Then $w(z)$ is a holomorphic function in \mathbb{L} if $\frac{\partial w}{\partial \bar{z}} = 0$, i.e.

$$u_x = v_y, \quad u_y = v_x.$$

Now we shall prove Theorem 2, which makes more precise the Weierstrass representation of maximal strongly regular space-like surfaces.

Proof of Theorem 2

Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y), (x, y) \in \mathcal{D} \subset \mathbb{L}$ be a maximal strongly regular space-like surface parameterized with canonical principal parameters. Since the Gauss map $l : \mathcal{D} \rightarrow H^2(1)$ of \mathcal{M} is conformal and the orthogonal frame field $l_x l_y l$ is negative oriented, then the vector function l is given locally by

$$(5.5) \quad \begin{aligned} \xi &= \frac{u^2(x, y) - v^2(x, y) - 1}{u^2(x, y) - v^2(x, y) + 1}, \\ l: \quad \eta &= \frac{2u(x, y)}{u^2(x, y) - v^2(x, y) + 1}, \quad u^2(x, y) - v^2(x, y) + 1 \neq 0, \\ \zeta &= \frac{2v(x, y)}{u^2(x, y) - v^2(x, y) + 1}, \end{aligned}$$

where $w = u(x, y) + j v(x, y)$ is a holomorphic function in \mathbb{L} .

We denote

$$\mu = \frac{u_x^2 - u_y^2}{(u^2 - v^2 + 1)^2}.$$

If ν is the normal curvature function of \mathcal{M} , then the vector function $\mathbf{z} = \mathbf{z}(x, y)$ satisfies the system

$$(5.6) \quad \begin{aligned} \mathbf{z}_x &= -\frac{1}{\nu} l_x = -\frac{1}{\nu} (u_x l_u + v_x l_v), \\ \mathbf{z}_y &= -\frac{1}{\nu} l_y = -\frac{1}{\nu} (u_y l_u + v_y l_v), \end{aligned}$$

which implies that $\nu = \frac{4(u_x^2 - u_y^2)}{(u^2 - v^2 + 1)^2} = 4\mu$. Hence the holomorphic function w satisfies the conditions

$$u^2 - v^2 + 1 \neq 0; \quad \mu > 0, \quad \mu_x \mu_y \neq 0.$$

Denoting by $w' = \frac{dw}{dz} = \frac{\partial w}{\partial z} = u_x + j u_y$, we have

$$\frac{1}{w'} = \frac{u_x - j u_y}{u_x^2 - u_y^2}.$$

Then (5.6) can be written in the form

$$\mathbf{z}_x + j \mathbf{z}_y = -\frac{1}{w'} \frac{(u^2 - v^2 + 1)^2}{4} (l_u - j l_v).$$

Taking into account (5.5), we obtain

$$\begin{aligned} (z_1)_x + j(z_1)_y &= -\frac{w}{w'} \\ (z_2)_x + j(z_2)_y &= \frac{1}{2} \frac{w^2 - 1}{w'}, \\ (z_3)_x + j(z_3)_y &= \frac{j}{2} \frac{w^2 + 1}{w'}, \end{aligned}$$

which proves the assertion. \square

As an application we obtain a corollary for the solutions of the natural partial differential equation.

Corollary 5.2. *Any solution ν of the natural partial differential equation (4.7) satisfying the condition (4.8) locally is given by the formula*

$$(5.8) \quad \nu = \frac{4(u_x^2 - u_y^2)}{(u^2 - v^2 + 1)^2}, \quad u^2 - v^2 + 1 \neq 0,$$

where $w = u + jv$ is a holomorphic function in \mathbb{L} .

Proof: Let $\nu(x, y)$ be a solution of (4.7) satisfying the conditions (4.8) and let us consider the minimal strongly regular space-like surface \mathcal{M} , generated by ν . According to Theorem 2 it follows that the normal curvature function ν of \mathcal{M} locally has the form (5.8). \square

It is a direct verification that any function ν , given by (5.8), where $w = u + jv$, ($u_x^2 - u_y^2 > 0$) is a holomorphic function in \mathbb{L} , satisfies (4.7).

Remark 5.3. The canonical Weierstrass representation of maximal strongly regular space-like surfaces is based on the Enneper space-like surface ($w = z$). Choosing any other minimal strongly regular space-like surface \mathcal{M} as a basic surface, we shall obtain its corresponding representation. This remark is also valid for the representation (5.8) of the solutions of the natural partial equation (4.7).

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